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PILE NEUTRON PHYSICS

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THE TRANSPORT KERNELS

It is possible to write the Boltzmann equation as an integral equation whenever the scattering and the source are isotropic. To do this we return to equation 1-50 (report M-3336) in which $F(x,v,\mu)$ is expressed as an integral over F_0 and S, the source distribution. Since we assume S is isotropic, we may put $S(x,v,\mu) = S_0(x,v)/2$: where S_0 is the total number of neutrons produced per cu cm per second. As it stands equation 1-50 is not quite an integral equation because the total $F(x,v)$ appears outside, and both functions are unknown. If, however, we integrate over μ, we then obtain an integral equation in $F_0(x,v)$:

$$F_0(x,v) = \int_{-1}^{1} F(x,v,\mu)\, d\mu = \frac{N\sigma_{S0}}{2} \int_0^1 \int_{-\infty}^{x} F_0(x',v)\, e^{-(N\sigma/\mu)(x-x')} \frac{dx'\, d\mu}{\mu} +$$

$$\frac{1}{\mu} \int_0^1 \int_{-\infty}^{x} S_0(x',v)\, e^{-(N\sigma/\mu)(x-x')} \frac{dx'}{\mu} + \frac{N\sigma_{S0}}{2} \int_{-1}^{0} \int_{\infty}^{x} F_0(x',v)\, e^{-(N\sigma/\mu)(x-x')} \frac{dx'}{\mu}\, d\mu +$$

$$\frac{1}{2} \int_{-1}^{0} \int_{\infty}^{x} S_0(x',v)\, e^{(N\sigma/\mu)(x-x')} \frac{dx'}{\mu}$$

and by interchanging order of integration,

$$F_0(x,v) = \frac{N\sigma_{S0}}{2} \int_{-\infty}^{\infty} F_0(x',v)\, E_1(N\sigma|x-x'|)\, dx' + \frac{1}{2} \int_{-\infty}^{\infty} S_0(x',v)\, E_1(N\sigma|x-x'|)\, dx' \qquad (1\text{-}151)$$

where

$$E_1(y) = \int_0^1 e^{-y/\mu} \frac{d\mu}{\mu} = \int_y^{\infty} e^{-\mu} \frac{d\mu}{\mu} \qquad (1\text{-}152)$$

is the exponential integral [denoted by $-Ei(-x)$ in Jahnke-Emde]. The total number of neutrons which start fresh flights per second in each cubic centimeter is $N\sigma_{S0}F_0(x,v) + S_0(x,v)$; this quantity, which we shall call $Q(x,v)$, may be viewed as the source which furnishes neutrons for the remainder of the medium. Hence the integral equation 1-51 (in report M-3336) may be written:

$$F_0(x,v) = \frac{1}{2} \int_{-\infty}^{\infty} Q(x',v)\, E_1(N\sigma|x-x'|)\, dx', \qquad (1\text{-}153)$$

$$Q(x',v) = N\sigma_{S0}F_0(x,v)\, S_0(x,v).$$

In this form the integral equation of isotropic transport theory resembles very much the integral equation 1-135 (in MDDC - 1023) of diffusion theory. However, the plane transport kernel [which we denote by $K_{pl}(x,x')$] is 1/2 $E_1(N\sigma|x-x'|)$, while the plane diffusion kernel is $1/(2\kappa D_0)\ e^{-\kappa|x-x'|}$. The two are related by the formula

$$K_{pl}(x,x') = D_0 \int_{N\sigma}^{\infty} G_{pl}(\kappa,x,x')\, d\kappa \tag{1-154}$$

The transport kernel gives the flux in a unit volume at r′ due to a unit source at r′.

The transport kernels in the other geometries can be obtained similarly from the diffusion kernels by integrating with respect to κ from $N\sigma$ to ∞ and multiplying by D_0. The results are tabulated in Table 1.

Table 1. Transport kernels.

Geometry	Notation	Source Normalization	K = flux at r′
Point	$K_p(\underline{r},\underline{r}')$	1 neut/sec	$\frac{1}{4\pi}\frac{e^{-N\sigma\|\underline{r}-\underline{r}'\|}}{\|r-r'\|^2}$
Plane	$K_{pl}(x,x')$	1 neut/sq cm/sec	$1/2\ E_0(N\sigma\|x-x'\|)$
Spherical Shell	$K_s(r,r')$	1 neut/shell/sec	$\frac{1}{8\pi rr'}\left\{E_0(N\sigma\|x-x'\|) - E_0(N\sigma\|r-r'\|)\right\}$
Line	$K_l(r,\phi,r',\phi)$	1 neut/cm/sec	$\frac{N\sigma}{2\pi}\int_1^{\infty} K_0(N\sigma\rho y)\, dy$, $\rho = \sqrt{r^2+r'^2\ 2rr'\ \cos(\phi-\phi)}$
Cylindrical Shell	$K_c(r,r')$	1 neut/cm/sec	$\frac{N\sigma}{2\pi}\int_1^{\infty} K_0(N\sigma ry)\, I_0(N\sigma r'y)\, dy,\ \ r > r'$ $\frac{N\sigma}{2\pi}\int_1^{\infty} K_0(N\sigma r'y)\, I_0(N\sigma ry)\, dy,\ \ r < r'$

The equivalence between a spherical shell and a point, or a cylindrical shell and a line, which hold for the diffusion and the potential kernels, does not hold for the transport kernels.

As an example of the use of a transport kernel we calculate, according to transport theory, the depression in neutron density caused by a thin foil which is introduced into an infinite medium in which monoenergetic neutrons are being produced everywhere at the constant rate q. This problem was treated by diffusion theory in a preceding paragraph, and here we use the same notation. The discussion which follows is in part due to Skyrme.

The integral equation for $F_0(\underline{r},v)$ in the absence of the foil, is

$$F_0(\underline{r},v) = \int_{\text{all space}} N\sigma_{S_0} F_0(\underline{r}',v)\, \frac{e^{-N\sigma|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|^2}\, dr' + \int_{\text{all space}} q\, \frac{e^{-N\sigma|\underline{r}-\underline{r}'|}}{4\pi|r-r'|^2}\, dr' \tag{1-155}$$

and this has the solution

$$F_0(\underline{r},v) = \frac{q}{N\sigma_a}, \qquad (1\text{-}156)$$

as can be verified by substitution into equation 1-155. If the foil is present, then it absorbs $(N\sigma_a)_F F_0$ neutrons per cu cm per second; this acts as a negative source. Further if we assume that the neutron makes no collisions in traversing the foil, except absorptions, so that the one-medium transport kernel is applicable, then the integral equation for F_0 must be

$$F_0(r,v) = \int_{\text{all space}} \left[N\sigma_{s_0}F_0(r,v)+q\right] \frac{e^{-N\sigma|\underline{r}-\underline{r}'|}}{4\pi|r-r'|^2}\,dr' - (N\sigma_a)_F \int_V F_0(r,v)\,\frac{e^{-N\sigma|r-r'|}}{4\pi|r-r'|^2}\,dr' \qquad (1\text{-}157)$$

Again, if the foil is so small that it hardly absorbs any neutrons, we can solve this equation by successive approximations, the first approximation being to put $F_0 = q/(N\sigma_a)$. The result is

$$F_0(\underline{r},v) = \frac{q}{N\sigma_a}\left(1-\frac{(N\sigma_a)_F}{4\pi}\int_V \frac{e^{-N\sigma|r-r'|}}{|r-r'|^2}\,dr'\right) \qquad (1\text{-}158)$$

while the diffusion theory result (we replace Φ_0 of equation 1-133b, in MDDC -1023, by F_0) is

$$F_0(\underline{r},v) = \frac{q}{N\sigma_a}\left(1-\frac{(N\sigma_a)_F}{4\pi D_0}\int_V \frac{e^{-N\sigma|\underline{r}-\underline{r}'|}}{|r-r'|}\,dr'\right)$$

If the foil is a small sphere of radius r_0, center at $r = 0$, these integrals are easily evaluated. The results for the density at the center of the sphere are

$$F_0(D,v) = \frac{q}{N\sigma_a}\left[1-\frac{(N\sigma_a)_F}{N\sigma}\left(1-e^{-N\sigma r_0}\right)\right] \text{ by transport theory} \qquad (1\text{-}159)$$

$$F_0(D,v) = \frac{q}{N\sigma_a}\left[1-\frac{3(N\sigma_a)_F}{N\sigma}\left(1-e^{-N\sigma r_0}(1+N\sigma r_0\right)\right] \text{ by diffusion theory.} \qquad (1\text{-}160)$$

Evidently, since the kernels used in this and in the diffusion calculation applied only to a single medium, the equations 1-159 and 1-160 will not give the depression in the interior of the foil correctly. This correction may be comparable to the depression outside the foil, especially if the foil absorbs neutrons heavily. We will calculate it later.

If the foil is so small that $N\sigma r_0 \ll 1$, then the depressions at the center are

$$\frac{F_0}{q/N\sigma_a} = 1-(N\sigma_a)_F\, r_0 \text{ by transport theory}$$

$$\frac{F_0}{q/N\sigma_a} = 1-3(N\sigma_a)_F\,(N\sigma)r_0^2 \text{ by diffusion theory.}$$

The difference between the two results is by no means negligible.

SOLUTION OF THE STEADY-STATE DIFFUSION EQUATION IN VARIOUS GEOMETRIES; MEASUREMENT OF DIFFUSION LENGTH

In this section we give a few examples of the calculation of the thermal neutron distribution in systems of particular shape and with certain source distributions.

We shall consider a rectangular parallelopiped of size $x = \bar{a}$, $y = \bar{b}$, $z = \infty$, with the source in the $z = 0$ plane, distributed like $f(x,y)$.

The neutron flux in this case is the solution of

$$\Delta \Phi_0 - \kappa^2 \Phi_0 = 0 \qquad (1\text{-}161)$$

with the initial condition

$$D_0 \frac{\partial \Phi_0}{\partial z}\bigg|_{z=0} = -1/2\ f(x,y), \qquad (1\text{-}162)$$

and the boundary conditions

$$\Phi_0 = 0,\ x = a,\ y = b, \qquad (1\text{-}163)$$

where a and b are the geometric sides, a, b, augmented by twice the extrapolation distance ($.71\lambda_{tr}$ in a weakly absorbing medium). The solution of equation 1-161 which satisfies all boundary and initial conditions is

$$\Phi_0 (x,y,z) = \frac{2}{D_0 a^2 b^2} \sum_n \sum_m f_{mn} L_{mn} \cos \frac{(2m+1)\pi x}{a} \cos \frac{(2n+1)\pi v}{b} e^{-z/L_{mn}} \qquad (1\text{-}164)$$

where

$$\frac{1}{L^2_{mn}} = K^2 + \frac{(2m+1)^2\pi^2}{a^2} + \frac{(2n+1)^2\pi^2}{b^2} \qquad (1\text{-}165)$$

and

$$f_{mn} = \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} f(x,y) \cos (2m+1) \frac{\pi y}{a} \cos (2n+1) \frac{\pi y}{b}\, dx\, dy \qquad (1\text{-}166)$$

The neutron flux falls off from the source as a sum of exponentials with relaxation lengths L_{mn} given by equation 1-165. The relaxation length of each harmonic decreases as the order m,n of the harmonic increases. Far from the source only the 0,0 harmonic remains.

The measurement of the diffusion length of a substance is usually done by measuring the distribution along the z direction of thermal neutrons in a block of the material in which a source of thermal neutrons has been placed. Since the actual distribution in the z direction is a sum of many harmonics, in order to deduce to relaxation length from the observed neutron distribution, it is necessary to take measurements far from the source, if intensity permits, or else to correct the observed relaxation length to the 0,0 relaxation length by subtracting the effect of the higher harmonics. The diffusion length is obtained from the observed L^2_∞ by equation 1-165; i.e.

$$\kappa^2 = \frac{1}{L^2} = \frac{1}{L^2_\infty} - \frac{\pi^2}{a^2}\ \frac{\pi^2}{b^2} \qquad (1\text{-}167)$$

If the absorber is weak the diffusion length is long. Hence unless the sides of the block are very large (π^2/a^2, $\pi^2/b^2 \ll 1/L_0^2$), the reciprocal diffusion length appears as the small difference of two relatively large numbers. In order to obtain results which are meaningful it is therefore necessary to measure L_∞, a, and b with extreme accuracy.

The technique which has generally been used on the Plutonium project for reducing the data in a diffusion length measurement has been the following:

1. From a knowledge of the source disposition the strength of the higher harmonics is estimated. These are subtracted from the observed neutron distribution to give the 0,0 harmonic. By a judicious choice of x,y coordinates for the neutron detectors it is possible to eliminate a few of the important harmonics. For example, in a square block, the (1,n) and (m,1) harmonic vanish at x = a/6, y = a/6.

2. Since the block is always of finite length, it is necessary to add an "end correction" to the observed intensities close to the end of the block. Suppose the extrapolated length of the block is z_0.

Then the neutron distribution which is zero at z and z_0 and satisfies equation 1-162 in the source plane is

$$\Phi_0(x,y,z) = \frac{2}{D_0 a^2 b^2} \sum_m \sum_n \frac{f_{mn} L_{mn}}{1+e^{-2z_0}} \left[e^{-z/L_{mn}} + e^{(z-2z_0)/L_{mn}} \right] \cos (2m+1) \frac{\pi x}{a} \cos \frac{(2n+1)\pi y}{b} \qquad (1\text{-}168)$$

The reflected "wave" $e^{(z-2z_0)/L_{mn}}$, which can be considered to arise from a negative image source in the plane $z = 2z_0$, is negligible compared to the incident wave, $e^{-z/L_{mn}}$, unless z is close to the extrapolated edge of the block. The end correction is made by subtracting the reflected wave, as estimated by the expression 1-168, from the observed distribution near the boundary.

3. After the harmonic and end corrections have been made it is customary to make a least squares fit to the longitudinal (z) distribution. The relaxation length of the best fitting exponential is used as L_∞ in equation 1-167, and from this L is determined.

4. Since the results are very sensitive to the values of a and b, transverse (x,y) neutron distributions are usually taken. If only one harmonic is present then the transverse distribution is strictly the product of two cosines, the half wave length of which are the extrapolated dimensions of the block. The strength of higher harmonics can be estimated from a harmonic analysis of the transverse distribution, although usually it is sufficiently accurate to compute these from a knowledge of the source distribution.

The most convenient source for a diffusion length measurement is a large block of graphite set on top of a chain reacting pile. Such a graphite block is called a thermal column, since neutrons from the pile are practically all reduced to thermal energy in the block, provided it is large enough. If the block whose diffusion length is to be measured is placed on top of the thermal column, then the neutrons impinging on it will all be thermal, and the theory outlined is immediately applicable.

Before chain reacting piles were available, thermal columns as neutron sources were impractical because neutron intensities were never high enough. Diffusion length measurements were performed by first measuring the thermal neutron distribution when an uncovered Ra-Be source (of fast neutrons) was in the source position, and then when the Ra-Be source was covered by a Cd sheet which absorbs all thermal neutrons. The difference between the thermal neutron densities in the two cases is just the thermal neutron distribution due to a source of pure thermal neutrons at the position of the Cd sheet. This can be seen by writing down the equations for the thermal neutron density in the two cases. Without the Cd sheet the flux Φ_0' satisfies

$$\Phi_0' - \kappa^2 \Phi_0' + \frac{q}{D_0} = 0$$

where q(x,z) is the number of neutrons which become thermal per second in the block. With the Cd sheet in place the flux Φ_0' satisfies the equation

$$\Delta\Phi_0'' - \kappa^2 \Phi_0'' + q/D_0 = 0$$

but with the boundary condition $\Phi_0'' = 0$ at z = 0, the extrapolated position of the Cd sheet. The difference $\Phi_0 = \Phi_0' - \Phi_0''$ satisfies

$$\Delta\Phi_0 - \kappa^2 \Phi_0 = 0$$

with the boundary condition

$$\Phi_0(x,y,0) = \Phi_0'(x,y,0),$$

where $\Phi_0(x,y,0)$ is the measured distribution at z = 0 without the Cd sheet.

THE TIME-DEPENDENT DIFFUSION EQUATION

The diffusion problems considered in the previous section have all been stationary problems. The neutron density was considered to be independent of time, and only the stationary spatial and angular distributions were sought. In this section we consider the nonstationary neutron diffusion problem from the elementary standpoint.

We consider an infinite plane system in which monoenergetic neutrons diffuse. According to elementary diffusion theory, the equation satisfied by n_0, the neutron density of speed v, is

$$D_0 \frac{vd^2n_0}{dx^2}(x,v,t) - N\sigma_a v n_0(x,v,t) + S_0(x,v,t) = \frac{\partial n_0}{\partial t}(x,v,t), \tag{1-169}$$

where $S_0(x,v,t)$ is the number of neutrons of speed v produced per cu cm per second at (x,t). Equation 1-169 is the same as the heat equation with leakage. Its solution for an instantaneous unit source at the origin, which emits one neutron per sq cm, $S_0(x,v,t) = \delta(x)\,\delta(t)$, is found by the usual Fourier transform method. Thus putting

$$\delta(x)\,\delta(t) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} e^{i\omega t}\, e^{i\nu x}\, d\omega\, d\nu, \quad n_0 = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} A(\omega,\nu)\, e^{i\omega t}\, e^{i\nu x}\, d\omega\, d\nu$$

and substituting into equation 1-169 we obtain

$$A = \frac{1}{D(\nu^2 + \kappa^2 + i\omega/D)},$$

where as usual $D = D_0 v, \kappa^2 = \frac{N\sigma_a v}{D}$; hence

$$n_0 = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \frac{e^{i\omega t}\, e^{i\nu x}}{D(\nu^2 + \kappa^2 + i\omega/D)}\, d\omega\, d\nu$$

This integral can be evaluated readily by integrating first over ω and then over ν. The integrand has a simple pole at $\omega = iD(\nu^2+\kappa^2)$, and its residue there is $(1/i)e^{-D(\nu^2+\kappa^2)t}$. Hence the integral over ν has the value $2\pi\, e^{-D(\nu^2+\kappa^2)t}$. The integral over ν is just the Fourier transform of a Gaussian function, and this is another Gaussian. Hence

$$n_0(x,v,t) = \frac{e^{-\frac{x^2}{4Dt} - \kappa^2 Dt}}{(4\pi Dt)^{1/2}}, \tag{1-170}$$

which is the well-known one dimensional nonsteady diffusion kernel. The properties of this function are very familiar since it also represents the temperature distribution from an instantaneous unit heat source. At any given time, the neutron distribution is Gaussian with a range $\sqrt{2Dt}$, and an amplitude $\frac{e^{-D\kappa^2 t}}{(4\pi Dt)^{1/2}}$. The attenuation factor $e^{-D\kappa^2 t}$ of course arises from the absorption by the medium. At any given point the neutron intensity waxes and wanes, reaching a maximum at time $\frac{\frac{1}{4}\kappa^2 x^2 - \frac{1}{2}}{2\kappa^2 D}$. If there is no absorption, the maximum is reached at time $t = \frac{x^2}{2D}$. For thermal neutrons diffusing in graphite, $\lambda = 2.7$ cm, $v = 2.2 \times 10^5$ cm/sec , $2D = 39.6 \times 10^4$ sq cm/sec , and the time for the neutron intensity to reach its maximum at a distance of 100 cm is about 25 milliseconds. Such a time lag is easily observable with standard electronic equipment.

According to equation 1-170, the effect of an instantaneous neutron source is felt everywhere immediately, although (except at the source) with small intensity. Evidently this cannot be quite correct

since a neutron burst requires at least the time x/v to travel the distance x from the source. During this "retarded time" no neutrons can appear at x. It will be shown later that the correct elementary theory time-dependent neutron-diffusion equation is really the equation of telegraphy (whose solutions are in fact retarded), rather than the equation 1-169 of heat conduction.

The retardation time is $t_{ret} = x/v$, while the time $t_{diffusion}$ for a neutron burst to reach a maximum at x is $x^2/2D$. The ratio of the two times is

$$\frac{t_{ret}}{t_{diffusion}} = \frac{2D}{xv} = \frac{2}{3}\frac{\lambda d}{x}.$$

At distances from the source that are large compared with a diffusion mean free path, the retarded time is negligible compared to the diffusion time. Since most experiments involve the neutron distribution far from the source, it is permissible to ignore the retardation and to describe time-dependent diffusion by means of the heat equation instead of the telegrapher's equation.

It is convenient at this point to give the nonstationary diffusion kernels, $G(\underline{r},t',\underline{r}',t)$ in various geometries (see Table 2). The method of deriving these kernels from the corresponding plane kernel is exactly the same as that used in a previous section for the steady-state diffusion kernels.

Table 2. Time-Dependent Diffusion Kernels

Notation	Normalization	Geometry	
$G_{pl}(x,t',x',t')$	1 neut/sq cm at (x',t')	plane	$\dfrac{e^{-\frac{\lvert x-x'\rvert^2}{4D(t-t')} - \kappa^2 D(t-t')}}{\left[4\pi D(t-t')\right]^{1/2}}$
$G_p(\underline{r},t',\underline{r}',t')$	1 neut at $(\underline{r}',t')$	point	$\dfrac{e^{-\frac{\lvert r-r'\rvert^2}{4D(t-t')} - \kappa^2 D(t-t')}}{\left[4\pi D(t-t')\right]^{3/2}}$
$G_l(r,\phi,t',r',\phi,t')$	1 neut/cm at (r',ϕ',t')	line	$\dfrac{e^{-\frac{\lvert \rho-\rho'\rvert^2}{4D(t-t')} - \kappa^2 D(t-t')}}{4\pi D(t-t')}$ $\rho^2 = r^2 + r'^2 - 2rr' \cos(\phi-\phi')$
$G_s(r,t',r',t')$	1 neut at r', per shell of radius at time t'	spherical shell	$\dfrac{e^{-\kappa^2 D(t-t')}}{4\pi rr'} \dfrac{e^{-\lvert r-r'\rvert^2/4D(t-t')}}{4\pi D(t-t')^{1/2}} - \dfrac{e^{\lvert r+r'\rvert^2/4D(t-t')}}{4\pi D(t-t')^{1/2}}$
$G_c(r,t',r',t')$	1 neut/ cm of shell r' at time t'	cylindrical shell	$\dfrac{e^{-\frac{(r-r')^2}{4D(t-t')}\kappa^2 D(t-t')}}{4\pi D(t-t')} I_0\left(\dfrac{rr'}{2D(t-t')}\right)$

It is useful to observe that the point, line and plane kernels differ only by powers of $\dfrac{1}{\left[4\pi D(t-t')\right]^{1/2}}$, and that the power of this quantity which appears in the kernel is just equal to the number of dimensions involved; eg., the point kernel (3 dimensions) has the factor $(4\pi D(t-t')^{3/2}$ in the denominator.

PROPAGATION OF NEUTRON WAVES

Suppose that instead of an instantaneous source of monoenergetic neutrons at x = 0, there is a localized source whose intensity oscillates with angular frequency ω:

$$S_0(x,v,t) = \delta(x)\, e^{i\omega t},$$

where it is understood here, and in the following, that only the real part of a complex function is to be used. The neutron intensity at (x,t) is found, exactly as in the previous paragraph to be

$$n_0(x,v,t) = \frac{e^{i\omega t}}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\nu x}\, d\nu}{D(\nu^2+\kappa^2+\frac{i\omega}{D})}\,.$$

The integrand has a pole above the real axis at $\nu = \pm\, v\sqrt{\kappa^2 + (i\omega/D)}$, and its residue there is just $\left(e^{-x\sqrt{\kappa^2+(i\omega/D)}}\right)\Big/\left(2Di\sqrt{\kappa^2+(i\omega/D)}\right)$. Hence the value of n_0, which is $2\pi i$ times the residue, is

$$n_0(x,v,t) = e^{i\omega t}\, \frac{e^{-x\sqrt{\kappa^2+(i\omega/D)}}}{2D\sqrt{\kappa^2+(i\omega/D)}} \qquad (1\text{-}171)$$

i.e., the distribution at time t is the same as from a stationary source of strength $e^{i\omega t}$, but the relaxation distance is the complex number $\sqrt{\kappa^2 + (i\omega/D)}$.

In order to understand the physical significance of the complex relaxation distances we write

$$\sqrt{\kappa^2+(i\omega/D)} = \sqrt{\frac{\kappa^2+\sqrt{\kappa^4+(\omega^2/D^2)}}{2}} + i\sqrt{\frac{-\kappa^2+\sqrt{\kappa^4+(\omega^2/D^2)}}{2}}\,.$$

Substituting into equation 1-171

$$n_0(x,v,t) = \frac{e^{i\left(\omega t-\sqrt{\frac{-\kappa^2+\rho^2}{2}}\,x\right)}\, e^{-\sqrt{\frac{\kappa^2+\rho^2}{2}}}}{SD\sqrt{\kappa^2+(i\omega/D)}} \qquad (1\text{-}172)$$

where

$$\rho^2 = \sqrt{\kappa^4 + (\omega/D)^2}$$

According to equation 1-172 the neutron density from an oscillating source is propagated as a damped wave; the velocity v_W of the wave is

$$v_W = \omega\sqrt{\frac{2}{\rho^2-\kappa^2}} \qquad (1\text{-}173)$$

its wave length l_W is

$$l_W = \frac{2\pi v}{\omega} = 2\pi\sqrt{\frac{2}{\rho^2-\kappa^2}} \qquad (1\text{-}174)$$

and its attenuation distance α, which is the distance over which its intensity falls by a factor e, is

$$\alpha = \sqrt{\frac{2}{\kappa^2+\rho^2}} \qquad (1\text{-}175)$$

The propagation velocity depends on the frequency, becoming larger as the frequency increases. Thus the medium in which the neutron waves travel is dispersive. The amplitude and the wave length of the wave fall off with increasing ω (cf. equation 1-172).

In the extreme cases $\frac{\omega}{D} \gg \kappa^2$ and $\frac{\omega}{D} \ll \kappa^2$ the formulas for the velocity, etc. become quite simple. We tabulate them below.

	$\frac{\omega}{D} \gg \kappa^2$	$\frac{\omega}{D} \ll \kappa^2$
v_W	$\sqrt{2D\omega/[1-(\kappa^2 D/\omega)]}$	$2\kappa D\left[1+\frac{1}{8}\left(\frac{\omega}{D\kappa^2}\right)^2\right]$
l_W	$2\pi\sqrt{\frac{2D}{\omega - \kappa^2 D}}$	$(4\pi\kappa D/\omega)\left[1+\frac{1}{8}\left(\frac{\omega}{D\kappa^2}\right)^2\right]$
α	$\sqrt{2(D/\omega)/[1+(\kappa^2 D/\omega)]}$	$\frac{1}{\kappa}\left[1-\frac{1}{8}\left(\frac{\omega}{D\kappa^2}\right)^2\right]$

The complete analogy between the propagation of neutron waves and the propagation of heat waves is evident from the foregoing discussion. The analogy was first pointed out by Wigner, who proposed that such basic constants as the thermal diffusion length, $1/\kappa$, and diffusion constant, D, could be determined by measuring the wave length and attenuation constant of neutron waves in a weakly absorbing medium. Such experiments would be completely analogous to the famous Angstrom method for measuring thermal conductivity. With the high neutron intensities available from a chain reacting pile such experiments should now be feasible.

We list below representative values of v_W, l_W, and α for thermal neutrons in graphite ($D_0 = 0.9$ cm, $v = 2.2 \times 10^5$ cm/sec, $\kappa = .02\ cm^{-1}$).

	$\omega = 10$/sec	$\omega = 1000$/sec
v_W	6×10^3 cm/sec	2×10^4 cm/sec
l_W	3.8×10^3 cm	126 cm
α	50 cm	20 cm

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